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EFFECT OF EDGE RESTRAINT ON THE BUCKLING OF AXIALLY COMPRESSED CIRCULAR CYLINDRICAL SHELLS

By

L. W. Rehfield

June 1970

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

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This program was carried out under Contract DA 44-177-AMC-258(T) with Stanford University.

The data contained in this report are the result of research conducted to study the effect of edge restraint on the buckling of circular cylindrical shells under axial compression. Sixteen limiting cases for edge restraint are studied, seven of them for the first time.

The report has been reviewed by the U.S. Army Aviation Materiel Laboratories and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

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EFFECT OF EDGE RESTRAINT ON THE BUCKLING OF AXIALLY COMPRESSED CIRCULAR CYLINDRICAL SHELLS

By

L. W. Rehfield

Prepared by

Department of Aeronautics and Astronautics
Stanford University
Stanford, California

for

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

SUMMARY

A broad analytical study of the effect of edge restraint on the buckling stress of geometrically perfect, axially compressed circular cylindrical shells, based on linear elastic theory, is presented. An overall evaluation of this effect is made by studying the 16 limiting cases of restraint which correspond to the vanishing of generalized forces and generalized displacements at an edge. Of the 16 limiting cases, 7 of which are studied for the first time, it is found that 8 permit buckling to occur at stresses less than the classical theoretical value of buckling stress. The fact that both theory and experiments indicate that the effect of shell length on the buckling stress can usually be disregarded is used in the analysis. This fact suggests that shells may be assumed to be semi-infinite; as a consequence, formulas for buckling stresses are determined in essentially closed form.

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LIST OF SYMBOLS

A,B	Constants appearing in equation (21)
$A_k (k=1-4)$	Constants appearing in equation (46)
C	Parameter defined in equation (26)
D	Flexural stiffness, $\frac{Et^3}{12(1-\nu^2)}$, lb-in.
E	Young's modulus of elasticity, psi
F	Airy stress function, lb
f	Dimensionless stress function defined in equation (13)
i	$+\sqrt{-1}$, the fundamental complex unit
k	Parameter defined in equation (71)
L	Cylinder length, in.
M_x, M_y, M_{xy}	Bending and twisting couple resultants, lb-in./in.
n	Circumferential wavelength parameter defined in equation (22)
$p_k (k=1-8)$	The roots of equation (24)
Q	Complex parameter defined in equation (28)
R	Mean radius of the circular cylindrical shell, in.
R_x	Effective transverse force resultant, lb/in.
r	Parameter defined in equation (41)
s	The number of circumferential waves in the buckle pattern
t	Uniform wall thickness of the shell, in.
U,V,W	Components of the displacement vector in the axial, tangential, and radial directions, respectively, in.
u,v,w	Dimensionless displacement functions defined in equation (13)
x,θ,z	Coordinates shown in the figure, in., rad., in., respectively
y	$= R\theta$, in.

$\alpha_k, \beta_k (k=1-2)$	Parameters defined in equation (44)
γ_{xy}^0	Shell midsurface shearing strain, in./in.
Δ	Symbolic determinant designation
$\epsilon_x^0, \epsilon_y^0$	Shell midsurface extensional strains, in./in.
$\kappa_x, \kappa_y, \kappa_{xy}$	Changes in curvature and twist of the shell midsurface, 1/in.
ν	Poisson's ratio
ξ, φ	Dimensionless coordinate parameters defined in equation (13)
ρ	$= \sigma/\sigma_{cl}$
σ	Mean axial compressive stress, psi
σ_{cl}	Classical critical value of σ, $\frac{E}{[3(1-\nu^2)]^{1/2}} \frac{t}{R}$, psi
$\sigma_x^0, \sigma_y^0, \tau_{xy}^0$	Membrane or direct stresses, psi

Subscripts

0	Denotes a basic or prebuckled state variable
1	Denotes a perturbation from the basic or prebuckled state value of a state variable

INTRODUCTION

Recent attention has been devoted to the study of the effect of edge restraint on the buckling behavior of thin circular cylindrical shells subjected to axial compression [1 through 6]*. A recent survey of the results of these studies and the conclusions drawn from them has been prepared by Hoff [7].

In addition to the strict enforcement of edge conditions at the instant of buckling, Stein [8,9], Fischer [10], and Almroth [11] have accounted for the prebuckling deformations due to edge restraint from the inception of compressive loading for shells with the various types of simply supported and completely clamped edges. The effect upon the compressive buckling stress of lateral pressure was also studied by these authors.

The present study is a broad treatment of the effect of edge restraint on the compressive buckling stress of circular cylindrical shells; the analysis is simplified by treating semi-infinite shells, since previous experience has indicated that the effect of shell length on the buckling stress is negligible, unless the shells are very short [5,6]. Limiting cases of edge restraint which correspond to the vanishing of generalized forces and generalized displacements at an edge are treated with the aid of the linear Donnell equations. The buckling solutions are obtained without great difficulty, and most of the results are presented in essentially closed form.

Of the 16 limiting cases of edge restraint, 7 are examined here for the first time; additional information regarding the remaining cases is also presented. In particular, a closed-form expression for the buckling criterion treated numerically in Reference 2 (and again in Reference 6) is derived, the solution found by Hoff in Reference 1 is extracted from this expression as a special case, and the results of Reference 6 for long shells with completely clamped edges are reestablished readily.

With these results at hand, it is possible to summarize the information obtainable from linear elastic theory regarding the effect of edge restraint on the buckling stress of axially compressed circular cylindrical shells. The governing equations which are utilized in the analysis are discussed thoroughly in order to put the results presented here and those previously obtained in References 2, 8, 9, 10, and 11 in proper perspective.

The present investigation represents a single detailed analysis in the overall program of plate and shell theoretical analyses currently in progress at Stanford University in the Department of Aeronautics and Astronautics. Unfortunately, there appears to be no published experimental data available at the present time with which the analytical results presented herein can be compared. A systematic, complementary, experimental investigation into edge-fixity effects is currently in progress under this same

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*Numbers in brackets indicate the references listed at the end of the report.

contract, but quantitative results are, as yet, unavailable. In view of the success of the statistical approach to shell buckling experiments developed earlier in the experimental phases of the contract work, reasonable correlation of test results with linear elastic theory can be expected.

BASIC THEORY

THE KÁRMÁN-DONNELL EQUATIONS

The plate equations of von Kármán were extended by Donnell [12] to the analysis of circular cylindrical shells; these equations are the foundation of the present analysis and may be written as

$$\nabla^4 F = E\left(W_{,xy}^2 - W_{,xx}W_{,yy} - \frac{W_{,xx}}{R}\right) \tag{1}$$

and

$$\left(\frac{D}{t}\right)\nabla^4 W - F_{,yy}W_{,xx} + 2F_{,xy}W_{,xy} - F_{,xx}\left(W_{,yy} + \frac{1}{R}\right) = 0 \tag{2}$$

where F is an Airy stress function defined such that

$$\sigma_x^0 = F_{,yy} \quad , \quad \tau_{xy}^0 = -F_{,xy} \quad , \quad \sigma_y^0 = F_{,xx} \tag{3}$$

Boundary conditions for complete cylinders with edges along constant x-curves are defined by the four following alternate choices of quantities which may be prescribed:

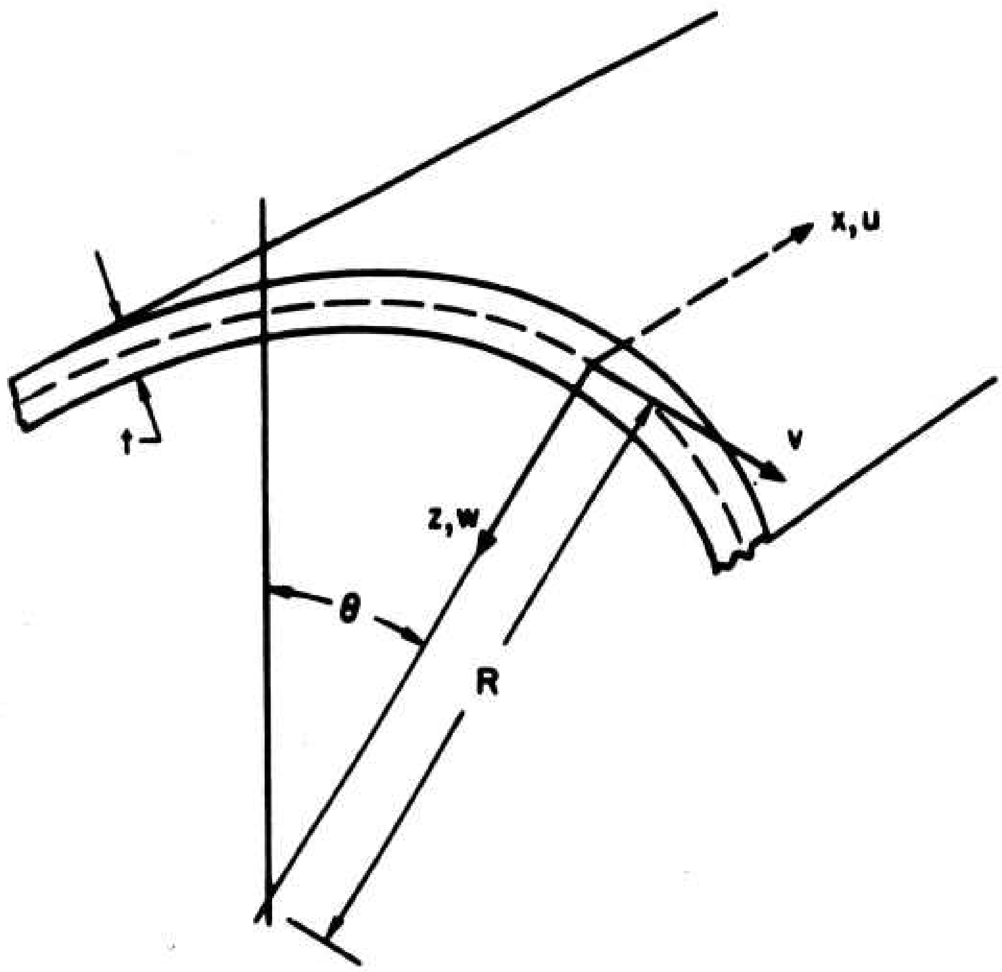
U	or	σ_x^0
V	or	τ_{xy}^0
W	or	$R_x = -D(W_{,xxx} + (2-\nu)W_{,xyy}) + t\sigma_x^0 W_{,x} + t\tau_{xy}^0 W_{,y}$
$W_{,x}$	or	$M_x = -D(W_{,xx} + \nu W_{,yy})$

With reference to the figure (page 3), U, V, and W are the components of the displacement vector in the axial, tangential, and radial directions, respectively, and the coordinates describing the undeformed shell midsurface are x, $y = R\theta$, and z.

The strains at the shell midsurface are given by

$$\epsilon_x^0 = U_{,x} + \frac{1}{2}(W_{,x})^2$$

(continued)



Circular Cylindrical Shell Coordinate System
and Sign Convention.

$$\left.\begin{aligned} \epsilon_y^0 &= V_{,y} - \frac{W}{R} + \frac{1}{2}(W_{,y})^2 \\ \gamma_{xy}^0 &= U_{,y} + V_{,x} + W_{,x}W_{,y} \end{aligned}\right\} \qquad (4)$$

The changes in curvature and twist of the shell midsurface are

$$\kappa_x = - W_{,xx} \quad , \quad \kappa_y = - W_{,yy} \quad , \quad \kappa_{xy} = - W_{,xy} \qquad (5)$$

The stress-strain and moment-curvature relations for an isotropic Hookean elastic material are

$$\left.\begin{aligned} E\epsilon_x^0 &= \sigma_x^0 - \nu\sigma_y^0 \\ E\epsilon_y^0 &= \sigma_y^0 - \nu\sigma_x^0 \\ \frac{E}{2(1+\nu)}\gamma_{xy}^0 &= \tau_{xy}^0 \end{aligned}\right\} \qquad (6)$$

$$\left.\begin{aligned} M_x &= D(\kappa_x + \nu\kappa_y) \\ M_y &= D(\kappa_y + \nu\kappa_x) \\ M_{xy} &= D(1-\nu)\kappa_{xy} \end{aligned}\right\} \qquad (7)$$

where M_x , M_y , M_{xy} are the moment resultants and $D = Et^3/12(1-\nu^2)$.

THE LINEARIZED DONNELL EQUATIONS FOR COMPRESSIVE AXIAL PRESTRESS

The membrane solution to the Kármán-Donnell equations for uniform axial compression is

$$\left.\begin{aligned} F_0 &= -\frac{\sigma y^2}{2} \\ W_0 &= -\frac{\nu\sigma R}{E} \\ U_0 &= -\frac{\sigma x}{E} \\ V_0 &= 0 \end{aligned}\right\} \qquad (8)$$

This membrane or "momentless" state can exist only if the uniform radial expansion W_0 is permitted to occur.

If equilibrium states which differ only slightly from the membrane state are considered and if sufficiently small changes in all quantities, say, F_1, U_1, V_1, and W_1, are assumed, the basic equations can be linearized. The substitution of $F_0 + F_1$, etc., into equation (1) through (7) and subsequent linearization result in the following equations governing the perturbations from the membrane state:

$$\nabla^4 F_1 = -\frac{E}{R} W_{1,xx} \tag{9}$$

and

$$\left(\frac{D}{t}\right)\nabla^4 W_1 + \sigma W_{1,xx} - \frac{F_{1,xx}}{R} = 0 \tag{10}$$

where the increments in the midsurface stresses are obtained from F_1 by the relations

$$\overset{0}{\sigma}_{x_1} = F_{1,yy} \quad , \quad \overset{0}{\tau}_{xy_1} = -F_{1,xy} \quad , \quad \overset{0}{\sigma}_{y_1} = F_{1,xx} \tag{11}$$

These stresses are related to the displacement components by the following equations:

$$\left.\begin{aligned} EU_{1,x} &= \overset{0}{\sigma}_{x_1} - \nu\overset{0}{\sigma}_{y_1} \\ E\left(V_{1,y} - \frac{W_1}{R}\right) &= \overset{0}{\sigma}_{y_1} - \nu\overset{0}{\sigma}_{x_1} \\ \frac{E}{2(1+\nu)}(U_{1,y} + V_{1,x}) &= \overset{0}{\tau}_{xy_1} \end{aligned}\right\} \tag{12}$$

Equations (9) through (12) have been put in a convenient nondimensional form by Nachbar and Hoff [2], who define the following new variables*:

$$u = \frac{U_1}{R}\left(\frac{2E}{\sigma_{cl}}\right)^{1/2} \qquad v = \frac{V_1}{R}\left(\frac{2E}{\sigma_{cl}}\right)^{1/2}$$

$$w = \frac{W_1}{R} \qquad f = \frac{F_1}{R^2 E}\left(\frac{2E}{\sigma_{cl}}\right) \qquad \rho = \frac{\sigma}{\sigma_{cl}} \tag{13}$$

$$\xi = \frac{x}{R}\left(\frac{2E}{\sigma_{cl}}\right)^{1/2} \qquad \varphi = \frac{y}{R}\left(\frac{2E}{\sigma_{cl}}\right)^{1/2}$$

*The notation used here is slightly different from the notation in reference 2.

where σ_{cl} is the classical buckling stress from linear theory given by the formula

$$\sigma_{cl} \quad \frac{E}{[3(1-\nu^2)]^{1/2}} \frac{t}{R} \tag{14}$$

With the aid of the definitions in equation (13), equations (9) through (12) can be replaced by the following dimensionless expressions:

$$\nabla^4 f + w_{,\xi\xi} = 0 \tag{15}$$

$$\nabla^4 w + 2\rho w_{,\xi\xi} - f_{,\xi\xi} = 0 \tag{16}$$

$$\sigma^0_{x_1} = Ef_{,\varphi\varphi} \,, \quad \tau^0_{xy_1} = -Ef_{,\xi\varphi} \,, \quad \sigma^0_{y_1} = Ef_{,\xi\xi} \tag{17}$$

$$\left.\begin{aligned} u_{,\xi} &= f_{,\varphi\varphi} - \nu f_{,\xi\xi} \\ v_{,\varphi} - w &= f_{,\xi\xi} - \nu f_{,\varphi\varphi} \\ u_{,\varphi} + v_{,\xi} &= -2(1+\nu) f_{,\xi\varphi} \end{aligned}\right\} \tag{18}$$

where $\nabla^2(\;) = (\;)_{,\xi\xi} + (\;)_{,\varphi\varphi}$ (19)

The boundary conditions for the linearized equations require the specification of

$$\begin{aligned} &u \quad \text{or} \quad \sigma^0_{x_1} = Ef_{,\varphi\varphi} \\ &v \quad \text{or} \quad \tau^0_{xy_1} = -Ef_{,\xi\varphi} \\ &w \quad \text{or} \quad R_x = -Et\left(\frac{\sigma_{cl}}{2E}\right)^{1/2}\left(w_{,\xi\xi\xi} + (2-\nu)w_{,\xi\varphi\varphi} + 2\rho w_{,\xi}\right) \end{aligned} \tag{20}$$

and

$$w_{,\xi} \quad \text{or} \quad M_x = -Rt\left(\frac{\sigma_{cl}}{2}\right)\left(w_{,\xi\xi} + \nu w_{,\varphi\varphi}\right)$$

Equations (15) through (18), (19), and (20) will be used in the subsequent buckling analysis; they are equivalent to the equations deduced by Donnell [13] for compressive axial prestress.

The purpose of describing in detail the derivation of the basic equations is to emphasize the following two points:

1. Consistent linear stability equations, along with their associated boundary conditions, are always obtained by linearizing consistent nonlinear equations about the basic state of equilibrium whose stability is to be investigated. The alternate techniques for deriving stability equations, even when based upon appropriate modification of small displacement theory by the inclusion of the effects of "apparent buckling forces", must be utilized with caution and are not recommended; the accuracy and consistency of the resulting equations and associated boundary conditions cannot be assured or easily assessed. The boundary conditions employed in Reference 2, for example, are inconsistent with the differential equations upon which the analysis is based,* as has been pointed out in Reference 6; a closed-form expression for the buckling criterion studied in Reference 2 will be derived subsequently, thus demonstrating the advantages of consistency.

2. The linear stability equations used by Stein [8,9], Fischer [10], and Almroth [11] in their studies of compressive buckling of cylindrical shells were derived by linearization of the Kármán-Donnell equations about basic states of equilibrium which differ from the membrane state equation (8). These authors account for the effect of axisymmetric prebuckling deformations, as previously mentioned in the introduction, which rigorously enforce the prescribed boundary conditions in the basic states of equilibrium. For axially compressed cylinders, Almroth [11], who also performed calculations with these nonuniform prebuckling deformations neglected, states that the results which he obtained using the equivalent of equations (9) and (10) are not significantly different from those obtained using equations in which the boundary conditions are rigorously enforced from the inception of compressive loading.

SOLUTIONS OF THE BASIC EQUATIONS FOR COMPLETE CYLINDRICAL SHELLS

THE FORM OF THE SOLUTIONS

In the case of the complete cylindrical shell, f and w must be periodic functions of the circumferential coordinate y. Since only derivatives of even orders appear in equations (15) and (16), it is obvious that products of functions of ξ alone with either $\sin(sy/R)$ or $\cos(sy/R)$, where s is an integer, can be made to satisfy them. The ordinary differential

*Although the problem was formulated with inconsistent boundary conditions, the numerical calculations were performed after the inconsistent terms had been discarded on the basis of an order-of-magnitude argument.

equations for functions of ξ which are obtained by such an assumed solution are solved by exponential functions of ξ; thus, without loss of generality, solutions are assumed of the form

$$\left.\begin{aligned} w &= Ae^{p\xi} \cos n\varphi \\ f &= Be^{p\xi} \cos n\varphi \end{aligned}\right\} \tag{21}$$

where A and B are constants; and, from equation (13), n must be related to s by the equation

$$n = s\left(\frac{\sigma_{cl}}{2E}\right)^{1/2} = s[12(1-\nu^2)]^{-1/4}\left(\frac{t}{R}\right)^{1/2} \tag{22}$$

By introducing equations (21) into equations (15) and (16), the homogeneous equations for the values of p are found to be

$$\begin{bmatrix} p^2 & (p^2-n^2)^2 \\ [(p^2-n^2)^2 + 2\rho p^2] & -p^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \tag{23}$$

For nontrivial solutions to equations (23), the determinant of the matrix of coefficients must vanish, which gives the characteristic equation

$$(p^2-n^2)^2[(p^2-n^2)^2 + 2\rho p^2] + p^4 = 0 \tag{24}$$

If $p^2 = n^2$, then only the trivial solution results; by excluding this possibility, equation (24) can be written as

$$C^2 + 2\rho C + 1 = 0 \tag{25}$$

where

$$C = \frac{(p^2-n^2)^2}{p^2} \tag{26}$$

From equations (23) it is found that

$$\frac{A}{B} = -C \tag{27}$$

where C is a solution to equation (25).

THE ROOTS OF EQUATION (24) for $\rho < 1$

Solutions to the characteristic equation (24) were first given by Nachbar [14] without the details of their derivation. A concise discussion of the solutions will be given here* for the purpose of providing various identities that will be used subsequently.

If $\rho \leq 1$, the two solutions of equation (25) can be conveniently written as

$$C_1 = -\rho + i(1-\rho^2)^{1/2} \equiv Q \qquad (28)$$

$$C_2 \quad -\rho - i(1-\rho^2)^{1/2} \equiv \bar{Q} \qquad (29)$$

where Q and $\bar{Q}$ are complex conjugates and $i = +\sqrt{-1}$. The modulus of Q is unity; that is,

$$Q\bar{Q} = 1 \qquad (30)$$

The following expressions for the positive square roots of Q and $\bar{Q}$ are noted:

$$Q^{1/2} = +\frac{1}{\sqrt{2}}[(1-\rho)^{1/2} + i(1+\rho)^{1/2}] \qquad (31)$$

$$\bar{Q}^{1/2} = +\frac{1}{\sqrt{2}}[(1-\rho)^{1/2} - i(1+\rho)^{1/2}] \qquad (32)$$

From equations (26), (28) and (29), four quadratic equations for the roots p are obtained; that is

$$\frac{p^2-n^2}{p} = \begin{bmatrix} - & Q^{1/2} \\ + & Q^{1/2} \\ - & \bar{Q}^{1/2} \\ + & \bar{Q}^{1/2} \end{bmatrix} \qquad \begin{matrix} (33) \\ (34) \\ (35) \\ (36) \end{matrix}$$

The two roots of equation (33), designated p_1 and p_5, are

$$p_1 = \frac{1}{2}\left[-Q^{1/2} - (Q+4n^2)^{1/2}\right] \qquad (37)$$

*The notation employed here differs from that of Reference 14; the notation of Reference 4 is utilized.

$$p_5 = \frac{1}{2}\left[-Q^{1/2} + (Q+4n^2)^{1/2}\right] \tag{38}$$

Similarly, the remaining six roots are obtained from equations (34), (35), and (36). They are

$$\begin{aligned}
&\text{from equation (34):} \quad && \left[\begin{aligned} p_2 &= -p_5 \\ p_6 &= -p_1 \end{aligned}\right. \\
&\text{from equation (35):} \quad && \left[\begin{aligned} p_3 &= \bar{p}_1 \\ p_7 &= \bar{p}_5 \end{aligned}\right. \\
&\text{from equation (36):} \quad && \left[\begin{aligned} p_4 &= -\bar{p}_5 \\ p_8 &= -\bar{p}_1 \end{aligned}\right.
\end{aligned} \tag{39}$$

The quantity $(Q+4n^2)^{1/2}$ is taken to be the positive square root of $(Q+4n^2)$ and may be written in the following way:

$$(Q+4n^2)^{1/2} = \frac{1}{\sqrt{2}}[(r+4n^2-\rho)^{1/2} + i(r-4n^2+\rho)^{1/2}] \tag{40}$$

where

$$r \equiv +(1-8n^2\rho + 16n^4)^{1/2} = +[(4n^2-\rho^2) + 1-\rho^2]^{1/2} \tag{41}$$

Note that if $\rho \leq 1$, then r must be positive.

With the aid of equations (31) and (40), p_1 and p_2 can be written with their real and imaginary parts explicitly separated; the expressions are

$$p_1 = \frac{1}{2\sqrt{2}}(\alpha_1 + i\beta_1) \tag{42}$$

and

$$p_2 = \frac{1}{2\sqrt{2}}(\alpha_2 + i\beta_2) \tag{43}$$

where

$$\alpha_1 = -(1-\rho)^{1/2} - (r+4n^2-\rho)^{1/2}; \qquad \beta_1 = -(1+\rho)^{1/2} - (r-4n^2+\rho)^{1/2}$$

$$\alpha_2 = (1-\rho)^{1/2} - (r+n^2-\rho)^{1/2}; \qquad \beta_2 = (1+\rho)^{1/2} - (r-4n^2+\rho)^{1/2} \tag{44}$$

The following identity is useful and may be verified by direct expansion:

$$(r+4n^2-\rho)^{1/2}(r-4n^2+\rho)^{1/2} = (1-\rho^2)^{1/2} \tag{45}$$

SOLUTIONS SUITABLE FOR ANALYZING THE BEHAVIOR OF LONG SHELLS

Four roots of equation (24), p_1 through p_4, have negative real parts for values of ρ less than unity; a proof of this statement is given in Appendix I. Roots of this type indicate that solutions which decay exponentially with ξ exist and that sufficiently long shells may be treated mathematically as semi-infinite whenever $\rho < 1$. A discussion of the applicability of "semi-infinite" buckling results to shells of finite length is given in the next section.

If a shell is assumed to be semi-infinite along the positive x-axis with its edge at $x = 0$, f and w will be bounded as $x \to \infty$ and can be written, with the aid of equations (27) through (30), (37), and (39), as follows when $\rho \leq 1$:

$$w = \left[A_1 e^{p_1\xi} + A_2 e^{p_2\xi} + A_3 e^{\bar{p}_1\xi} + A_4 e^{\bar{p}_2\xi}\right] \cos n\varphi \tag{46}$$

$$f = -\left[A_1 Q^{-1} e^{p_1\xi} + A_2 Q^{-1} e^{p_2\xi} + A_3 \bar{Q}^{-1} e^{\bar{p}_1\xi} + A_4 \bar{Q}^{-1} e^{\bar{p}_2\xi}\right] \cos n\varphi$$

$$= -\left[A_1 \bar{Q} e^{p_1\xi} + A_2 \bar{Q} e^{p_2\xi} + A_3 Q e^{\bar{p}_1\xi} + A_4 Q e^{\bar{p}_2\xi}\right] \cos n\varphi \tag{47}$$

where A_1 through A_4 are arbitrary constants. Substituting equation (47) into equation (17) yields the following results:

$$\frac{\sigma^0_{x_1}}{E} = f_{,\varphi\varphi} = n^2\left[A_1 \bar{Q} e^{p_1\xi} + A_2 \bar{Q} e^{p_2\xi} + A_3 Q e^{\bar{p}_1\xi} + A_4 Q e^{\bar{p}_2\xi}\right] \cos n\varphi \tag{48}$$

$$\frac{\tau^0_{xy_1}}{E} = -f_{,\xi\varphi} = -n\left[A_1 p_1 \bar{Q} e^{p_1\xi} + A_2 p_2 \bar{Q} e^{p_2\xi} + A_3 \bar{p}_1 Q e^{\bar{p}_1\xi} + A_4 \bar{p}_2 Q e^{\bar{p}_2\xi}\right] \sin n\varphi \tag{49}$$

$$\frac{\sigma^0_{y_1}}{E} = f_{,\xi\xi} = -\left[A_1 p_1^2 \bar{Q} e^{p_1\xi} + A_2 p_2^2 \bar{Q} e^{p_2\xi} + A_3 \bar{p}_1^2 Q e^{\bar{p}_1\xi} + A_4 \bar{p}_2^2 Q e^{\bar{p}_2\xi}\right] \cos n\varphi \tag{50}$$

Substituting equations (46), (48), and (50) into the expression for $v_{,\varphi}$ in equations (18) gives

$$v_{,\varphi} = f_{,\xi\xi} - \nu f_{,\varphi\varphi} + w$$

$$= \Big[A_1[1-\bar{Q}(p_1^2+\nu n^2)]e^{p_1\xi} + A_2[1-\bar{Q}(p_2^2+\nu n^2)]e^{p_2\xi}$$

$$+ A_3[1-Q(\bar{p}_1^2 + \nu n^2)]e^{\bar{p}_1\xi} + A_4[1-Q(\bar{p}_2^2 + \nu n^2)] \Big] \cos n\varphi \qquad (51)$$

The utilization of equations (48) and (50) in the expression for $u_{,\xi}$, equations (18), and subsequent integration of the result yields

$$u = \Big[A_1\bar{Q} \frac{(n^2+\nu p_1^2)}{p_1} e^{p_1\xi} + A_2\bar{Q} \frac{(n^2+\nu p_2^2)}{p_2} e^{p_2\xi}$$

$$+ A_3Q \frac{(n^2+\nu\bar{p}_1^2)}{\bar{p}_1} e^{\bar{p}_1\xi} + A_4Q \frac{(n^2+\nu\bar{p}_2^2)}{\bar{p}_2} e^{\bar{p}_2\xi} \Big] \cos n\varphi \qquad (52)$$

The process of integration with respect to ξ could introduce a linear function of ξ in addition to the terms given in equation (52). This linear function, however, represents the same form of displacement as U_0, and thus it is uniquely included in U_0 when the axial stress σ is fixed in magnitude.

Equation (52) can be put in a more convenient form by the use of the identity

$$\frac{p_1p_2}{n^2} = \frac{\bar{p}_1\bar{p}_2}{n^2} = 1 \qquad (53)$$

which may be verified by direct expansion. By using this result, equation (52) can be written as

$$u = [A_1\bar{Q}(p_2+\nu p_1)e^{p_1\xi} + A_2\bar{Q}(p_1+\nu p_2)e^{p_2\xi}$$

$$+ A_3Q(\bar{p}_2+\nu\bar{p}_1)e^{\bar{p}_1\xi} + A_4Q(\bar{p}_1+\nu\bar{p}_2)e^{\bar{p}_2\xi}] \cos n\varphi \qquad (54)$$

All the necessary physical quantities can now be obtained from equations (46), (48) through (51), and (54) for the semi-infinite shell in terms of the four arbitrary constants A_1 through A_4.

APPLICABILITY OF SEMI-INFINITE SHELL BUCKLING SOLUTIONS TO SHELLS OF FINITE LENGTH

In the preceding discussion of solutions to the governing differential equations, the solution suitable for the analysis of semi-infinite shells was assumed to be adequate for "long" shells. A shell is "long" according to this definition if the conditions at one of its edges can be analyzed without regard to the conditions prevailing at the other.

It is well known that there are two decay lenghts for edge disturbances in circular cylindrical shells; these correspond to the exponentials proportional to α_1 and α_2. Since the quantity n is substantially less than unity for thin shells, it follows from the proof in Appendix I that the α_2 - exponential terms in the semi-infinite shell solution have large decay lengths, thus suggesting that shells may have to be very long before this solution alone can satisfactorily describe their buckling behavior.

Results obtained in References 5 and 6, however, indicate that buckling stresses are essentially independent of shell length for shells of practical geometric proportions in the cases of edge restraint that were investigated*; buckling stresses obtained in References 2 and 4 for semi-infinite shells agree well with their finite-length counterparts for all but the very short class of shells. The buckling mode shapes in some cases offer intuitive justification for this agreement between "finite-length" and "semi-infinite" buckling stresses. The modes for finite-length shells found in Reference 5, for example, can be visualized as having been constructed by appropriately joining, in a continuous manner, two oppositely directed, semi-infinite modes at the midlength cross section of the cylinder; the superposition of a rigid body rotation and a reflection of one of the semi-infinite modes could represent the finite-length mode which is antisymmetric with respect to the midlength cross section.

Since past experience points to the conclusion that semi-infinite shell analyses, which offer great mathematical simplicity, predict nearly the same buckling stresses as their more involved finite-length counterparts, the semi-infinite assumption will be used exclusively in the subsequent determination of buckling stresses. Little generality is sacrificed in adopting this assumption, since very short shells, the class of shells for which this assumption has been shown to be in error in Reference 6, are likely to buckle well into the plastic range.

*There is one exceptional case that was found in Reference 6. This is the case of a relatively short, completely unrestrained cylindrical shell, and it will be discussed in a subsequent section.

CLASSIFICATION OF EDGE CONDITIONS

SCOPE OF THE INVESTIGATION

When cylindrical shells are utilized as structural components of flight vehicles, their edges are restrained by adjacent structural components, which may be represented by "effective" elastic or inelastic springs. When cylindrical shells are tested in compression under laboratory conditions, loads may also be supplied to the shell edges by friction between the source of load application and the shell or its mounting fixtures. Such effects will not be considered here; the types of edge fixity to be treated may be regarded as limiting cases of more realistic edge-fixity conditions.

The edge conditions characterized by equations (20) can be conveniently separated into two categories which, for the lack of better terminology, will be referred to as "normal" and "tangential" edge conditions. Normal edge conditions refer to the conditions involving quantities that govern the motion of the shell edge in the direction normal to the tangent plane of the shell midsurface at any point; these are then conditions describing the behavior of an edge due to bending and twisting. Tangential edge conditions, on the other hand, govern the motion of the shell edge in the tangent plane to the shell midsurface at any point and characterize the behavior of the edge under loads parallel to the tangent plane. Since only limiting cases of practical edge restraint are to be studied here, these two categories will be classified separately.

CLASSIFICATION OF NORMAL EDGE CONDITIONS

Each of the following four classes of normal edge conditions will be identified by two letters which are abbreviations for the terminology assigned to the particular class:

FF: The "free-free" class for which

$$R_x = 0 \quad \text{and} \quad M_x = 0 \tag{55}$$

FC: the "free-clamped" class for which

$$R_x = 0 \quad \text{and} \quad w_{,\xi} = 0 \tag{56}$$

SS: The "simply supported" class for which

$$w = 0 \quad \text{and} \quad M_x = 0 \tag{57}$$

CC: The "completely clamped"* class for which

$$w = 0 \quad \text{and} \quad w_{,\xi} = 0 \tag{58}$$

It should be mentioned that within the linearized theory of static stability, the condition $w = 0$ in equations (57) and (58) may be replaced by

$$W = W_0 + W_1 = 0 \tag{59}$$

If the perturbation displacement W_1 is chosen to nullify the uniform radial expansion W_0, then the displacement W_1 will approach infinity at the buckling stress. Each of these equivalent ways of prescribing the boundary conditions leads to the same unique buckling stress, since the buckling condition is defined by the vanishing of the same determinant in both instances.

CLASSIFICATION OF TANGENTIAL EDGE CONDITIONS

Each of the four classes of tangential edge conditions will be identified by an integer; the selection of one class of normal edge conditions together with one of the following classes of tangential edge conditions represents a complete description of edge restraint:

$$1: \quad \sigma^0_{x_1} = 0 \quad \text{and} \quad \tau^0_{xy_1} = 0 \tag{60}$$

$$2: \quad u = 0 \quad \text{and} \quad \tau^0_{xy_1} = 0 \tag{61}$$

$$3: \quad \sigma^0_{x_1} = 0 \quad \text{and} \quad v = 0 \tag{62}$$

$$4: \quad u = 0 \quad \text{and} \quad v = 0 \tag{63}$$

As an example the classical buckling stress is obtained for SS3 edge restraint when equations (57) and (62) are enforced at the ends of the shell.

*"Clamped" is a conventional description of this class; the term "completely clamped" is introduced to provide a contrast with the "free-clamped" definition given previously. No standard terminology has been agreed upon, however; the authors of Reference 6, for example, refer to this class of restraint as "rigid-fixed".

BUCKLING STRESS RATIOS FOR LIMITING CASES OF EDGE RESTRAINT

FF1 CASE

This case was treated initially by Hoff [1] for axisymmetric buckling deformations. The generalization to nonaxisymmetric buckling was subsequently accomplished by Nachbar and Hoff [2]; in this study, however, boundary conditions which were inconsistent with the differential equations that the authors utilized in the analysis were employed to formulate the problem. This inconsistency was later pointed out by Hoff and Soong [6] and was corrected in their extended analysis which accounted for the effect of finite shell length.

Although the correct numerical result for long shells was essentially obtained in Reference 2 and was verified in Reference 6, it is of theoretical interest to reinvestigate this case, for it will subsequently be shown that a closed-form expression for the buckling criterion is obtainable. However, the important result of the analysis is the procedure of simplification which can be used directly to treat other cases without great difficulty. A collection of identities which will be used in the following analysis appears in Appendix II.

With the aid of equations (46), (48), (49), and (20), the requirements given in equations (55) and (60) for the edge at x (or ξ) = 0 lead to the following four algebraic equations:

$$\begin{bmatrix} \bar{Q} & \bar{Q} & Q & Q \\ p_1\bar{Q} & p_2\bar{Q} & \bar{p}_1 Q & \bar{p}_2 Q \\ (p_1^2-\nu n^2) & (p_2^2-\nu n^2) & (\bar{p}_1^2-\nu n^2) & (\bar{p}_2^2-\nu n^2) \\ p_1[p_1^2+2\rho-(2-\nu)n^2] & p_2[p_2^2+2\rho-(2-\nu)n^2] & \bar{p}_1[\bar{p}_1^2+2\rho-(2-\nu)n^2] & \bar{p}_2[\bar{p}_2^2+2\rho-(2-\nu)n^2] \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0 \qquad (64)$$

The rows of the matrix of coefficients correspond to the vanishing of $\sigma^0_{x_1}$, $\tau^0_{xy_1}$, M_x, and R_x, respectively.

Equation (64) represents a system of homogeneous, linear algebraic equations; nontrivial solutions for the constants A_1 through A_4 are possible only if the determinant of the matrix of coefficients vanishes. The value of the stress ratio ρ for which the determinant vanishes are critical values which correspond to buckling within the scope of linear static stability theory.

Since the vanishing of the determinant of the matrix of coefficients is the only factor of interest here, it is desirable to multiply the rows and columns of the determinant by nonzero quantities in order to obtain algebraic simplifications. The first operations to be performed are:

1. Multiply the columns of the matrix by p_2, p_1, $\bar{p}_2$, and $\bar{p}_1$, respectively; and in view of equation (53), n^2 factors may be disregarded in the last three rows of the matrix since they do not influence the vanishing of the determinant.

2. Subtract the first column from the second, and subtract the third column from the fourth.

The modified determinantal equation resulting from the above operations is

$$\begin{vmatrix} p_2\bar{Q} & \bar{Q}(p_1-p_2) & \bar{p}_2Q & Q(\bar{p}_1-\bar{p}_2) \\ \bar{Q} & 0 & Q & 0 \\ (p_1-\nu p_2) & (1+\nu)(p_2-p_1) & (\bar{p}_1-\nu\bar{p}_2) & (1+\nu)(\bar{p}_2-\bar{p}_1) \\ p_1^2+[2\rho-(2-\nu)n^2] & (p_2^2-p_1^2) & \bar{p}_1^2+[2\rho-(2-\nu)n^2] & (\bar{p}_2^2-\bar{p}_1^2) \end{vmatrix} = 0 \tag{65}$$

The factor (p_2-p_1) multiplying column 2 and the factor $(\bar{p}_2-\bar{p}_1)$ may be removed, since they represent multiplication of the remaining determinant by

$$(p_2-p_1)(\bar{p}_2-\bar{p}_1) = 1 \tag{66}$$

Thus, equation (65) can be written as

$$\begin{vmatrix} p_2\bar{Q} & -\bar{Q} & \bar{p}_2Q & -Q \\ \bar{Q} & 0 & Q & 0 \\ (p_1-\nu p_2) & (1+\nu) & (\bar{p}_1-\nu\bar{p}_2) & (1+\nu) \\ p_1^2+[2\rho-(2-\nu)n^2] & (p_1+p_2) & \bar{p}_1^2+[2\rho-(2-\nu)n^2] & (\bar{p}_1+\bar{p}_2) \end{vmatrix} = 0 \tag{67}$$

The following operations are now performed:

1. Multiply column 2 by p_2 and add the result to column 1; multiply column 4 by $\bar{p}_2$ and add the result to column 3.
2. Equation (53) and the following identities are used to simplify the modified first and third columns:

$$p_1^2 + p_2^2 = 2n^2 + Q \tag{68}$$

$$\bar{p}_1^2 + \bar{p}_2^2 = 2n^2 + \bar{Q} \tag{69}$$

The determinantal equation becomes

$$\begin{vmatrix} 0 & -\bar{Q} & 0 & -Q \\ \bar{Q} & 0 & Q & 0 \\ (p_1+p_2) & (1+\nu) & (\bar{p}_1+\bar{p}_2) & (1+\nu) \\ Q+k & (p_1+p_2) & \bar{Q}+k & (\bar{p}_1+\bar{p}_2) \end{vmatrix} = 0 \tag{70}$$

where

$$k = 2\rho + (1+\nu)n^2 \tag{71}$$

If the rows and columns of the determinant in equations (70) are rearranged, the result can be conveniently written as the sum of two determinants

$$\Delta = \Delta_1 + \Delta_2 = 0 \tag{72}$$

where

$$\Delta_1 = \begin{vmatrix} \bar{Q} & Q & 0 & 0 \\ Q & \bar{Q} & 0 & 0 \\ 0 & 0 & -Q & -\bar{Q} \\ (p_1+p_2) & (\bar{p}_1+\bar{p}_2) & (1+\nu) & (1+\nu) \end{vmatrix} = (1+\nu)(\bar{Q}-Q)^2(Q+\bar{Q}) \tag{73}$$

and

$$\Delta_2 = \begin{vmatrix} Q & Q & 0 & 0 \\ 0 & 0 & -Q & -\bar{Q} \\ (p_1+p_2) & (\bar{p}_1+\bar{p}_2) & (1+\nu) & (1+\nu) \\ k & k & (\bar{p}_1+\bar{p}_2) & (p_1+p_2) \end{vmatrix} = k(1+\nu)(\bar{Q}-Q)^2-[(\bar{p}_1+\bar{p}_2)\bar{Q}-(p_1+p_2)Q]^2 \tag{74}$$

The substitution of equations (73) and (74) into equation (72) yields the simple result

$$(\bar{Q}-Q)^2(1+\nu)[(Q+\bar{Q}) + k] = [(\bar{p}_1+\bar{p}_2)\bar{Q} - (p_1+p_2)Q]^2 \tag{75}$$

However,

$$Q + \bar{Q} + k = (1+\nu)n^2 \tag{76}$$

so equation (75) offers the two possible solutions

$$(\bar{Q}-Q)(1+\nu)n = \pm[(\bar{p}_1+\bar{p}_2)\bar{Q} - (p_1+p_2)Q] \tag{77}$$

By direct expansion, the following expressions are easily obtained:

$$(\bar{p}_1+\bar{p}_2)\bar{Q} - (p_1+p_2)Q = -\frac{2i}{\sqrt{2}}[\rho(r-4n^2+\rho)^{1/2} -(1-\rho^2)^{1/2}(r+4n^2-\rho)^{1/2}] \tag{78}$$

$$(\bar{Q}-Q) = -2i(1-\rho^2)^{1/2} \tag{79}$$

Equation (77) is, therefore, equivalent to

$$(1-\rho^2)^{1/2}\sqrt{2}(1+\nu)n = \pm[\rho(r-4n^2+\rho)^{1/2} - (1-\rho^2)^{1/2}(r+4n^2-\rho)^{1/2}] \tag{80}$$

The quantity $(r+4n^2-\rho)^{1/2}$ cannot vanish unless $\rho = 1$. It may also be observed that $\rho = 1$ will satisfy equation (80). Since solutions for $\rho < 1$ are of interest, the multiplication of equation (80) by $(r+4n^2-\rho)^{1/2}$ is permissible; and, if the common factor $(1-\rho^2)^{1/2}$ is ignored after equation (45) is employed, the following two alternatives are deduced from equation (80):

$$\sqrt{2}\,(1+\nu)n(r+4n^2-\rho)^{1/2} = \rho-(r+4n^2-\rho) \tag{81}$$

$$\sqrt{2}\,(1+\nu)n(r+4n^2-\rho)^{1/2} = -\rho+(r+4n^2-\rho) \tag{82}$$

The smallest value of ρ is obtained from equation (82); it was found numerically in Reference 6 to be 0.380 for $\nu = 0.3$ and corresponds to $n = 0.30$.

The above equations are extremely sensitive to approximate techniques of solution. However, if n is permitted to vanish in either equation (81) or (82), which implies axisymmetric deformation, the solution $\rho = 1/2$ results; this is the solution that was first obtained by Hoff in Reference 1.

In their study of finite-length effects, Hoff and Soong [6] discovered that very short shells with FF1 edges can be buckled by extremely low values of stress; the mode of instability is one of twisting as a curved strip and can easily be visualized. This mode of buckling, however, should not be a matter of concern for shells that are currently of interest.

FF2 CASE

Equations (55) and (61) will be satisfied at $\xi = 0$ if the first row of the coefficient matrix in equations (64) is replaced by

$$[p_2\bar{Q} \quad p_1\bar{Q} \quad \bar{p}_2 Q \quad \bar{p}_1 Q] \tag{83}$$

If the same sequence of operations is performed which simplified the FF1 case to the form given in equations (70), the buckling condition for an FF2-supported shell can be written

$$\begin{vmatrix} -\bar{Q}n^2 & -\bar{Q}(p_1+p_2) & -Qn^2 & -Q(\bar{p}_1+\bar{p}_2) \\ Q & 0 & Q & 0 \\ (p_1+p_2) & (1+\nu) & (\bar{p}_1+\bar{p}_2) & (1+\nu) \\ Q+k & (p_1+p_2) & \bar{Q}+k & (\bar{p}_1+\bar{p}_2) \end{vmatrix} = 0 \tag{84}$$

If row 2 is multiplied by n^2 and added to row 1 in equation (84) and the rows and columns are rearranged, the buckling condition can be written

$$\Delta = \Delta_1 + \Delta_2 = 0 \tag{85}$$

where

$$\Delta_1 = \begin{vmatrix} \bar{Q} & Q & 0 & 0 \\ Q & \bar{Q} & 0 & 0 \\ 0 & 0 & -Q(\bar{p}_1+\bar{p}_2) & -\bar{Q}(p_1+p_2) \\ (p_1+p_2) & (p_1+p_2) & (1+\nu) & (1+\nu) \end{vmatrix} = (1+\nu)(\bar{Q}^2-Q^2)[\bar{Q}(p_1+p_2) - Q(\bar{p}_1+\bar{p}_2)] \tag{86}$$

and

$$\Delta_2 = \begin{vmatrix} \bar{Q} & Q & 0 & 0 \\ 0 & 0 & -Q(\bar{p}_1+\bar{p}_2) & -\bar{Q}(p_1+p_2) \\ (p_1+p_2) & (p_1+p_2) & (1+\nu) & (1+\nu) \\ k & k & (\bar{p}_1+\bar{p}_2) & (p_1+p_2) \end{vmatrix}$$

or

$$\Delta_2 = (\bar{Q}-Q)k(1+\nu)[\bar{Q}(p_1+p_2)-Q(\bar{p}_1+\bar{p}_2)]-(\bar{Q}-Q)r[\bar{Q}(\bar{p}_1+\bar{p}_2)-Q(p_1+p_2)] \tag{87}$$

by use of the identity

$$(p_1+p_2)(\bar{p}_1+\bar{p}_2) = r \tag{88}$$

The substitution of equations (86) and (87) into (85), along with the use of equation (76), gives

$$(\bar{Q}-Q)\left\{(1+\nu)^2n^2[\bar{Q}(p_1+p_2)-Q(\bar{p}_1+\bar{p}_2)]-r[\bar{Q}(\bar{p}_1+\bar{p}_2)-Q(p_1+p_2)]\right\} = 0 \tag{89}$$

The quantity $[(\bar{p}_1+\bar{p}_2)\bar{Q}-(p_1+p_2)Q]$ is given in equation (78), and direct expansion of the other terms leads to

$$[\bar{Q}(p_1+p_2)-Q(\bar{p}_1+\bar{p}_2)] = \frac{2i}{\sqrt{2}}[\rho(r-4n^2+\rho)^{1/2}+(1-\rho^2)^{1/2}(r+4n^2-\rho)^{1/2}] \tag{90}$$

If equation (89) is multiplied by $(r+4n^2-\rho)^{1/2}$ and constant terms and factors which vanish at $\rho = 1$ are disregarded, the buckling condition becomes

$$[r+(1+\nu)^2n^2]\rho = [r-(1+\nu)^2n^2](r+4n^2-\rho) \tag{91}$$

or

$$r^2 +[(4-(1+\nu)^2)n^2 - 2\rho]r - 4(1+\nu)^2n^4 = 0 \tag{92}$$

The quantity n^2 is proportional to t/R, and thus it is small compared to unity for thin shells. By neglecting the n^4 term in equation (92) and by noting that r can vanish only for $\rho > 1$, an approximate solution to equation (92) is found to be

$$r \approx 2\rho -[4-(1+\nu)^2]n^2 \tag{93}$$

If terms in n^4 and higher powers of n are neglected in the definition of r, the following approximate expression results:

$$r \approx 1 - 4n^2\rho \tag{94}$$

Equations (93) and (94) yield the approximate buckling stress ratio

$$\rho \approx \frac{1}{2}\{1+[2-(1+\nu)^2]n^2\} \tag{95}$$

which is very nearly one-half for thin shells.

FF3 AND FF4 CASES

These cases of edge restraint involve the replacement of the condition $\tau^0_{xy_1} = 0$ with $v = 0$ in the two preceding analyses; the second row of the coefficient matrix in equations (64) is replaced by

$$\{[1-\bar{Q}(p_1^2+\nu n^2)][1-\bar{Q}(p_2^2+\nu n^2)][1-Q(\bar{p}_1^2+\nu n^2)][1-Q(\bar{p}_2^2+\nu n^2)]\} \tag{96}$$

The systematic simplification procedure employed in the two previous cases can be utilized again. The algebra is slightly more involved, but the buckling stress ratios for both FF3 and FF4 restraints are one-half plus small correction terms, the first of which is proportional to n^2 as in the FF2 case.

FC1 CASE

The simultaneous satisfaction of equations (56) and (60) for nontrivial values of A_1 through A_4 is assured if the following determinantal equation is satisfied:

$$\begin{vmatrix} \bar{Q} & \bar{Q} & Q & Q \\ p_1\bar{Q} & p_2\bar{Q} & \bar{p}_1 Q & \bar{p}_2 Q \\ p_1 & p_2 & \bar{p}_1 & \bar{p}_2 \\ p_1^3 & p_2^3 & \bar{p}_1^3 & \bar{p}_2^3 \end{vmatrix} = 0 \tag{97}$$

If the following operations are performed on the above determinant,

1. Multiply the columns of the determinant by p_2, p_1, $\bar{p}_2$, and $\bar{p}_1$, respectively, and disregard the n^2 factor appearing in the last three rows.

2. Subtract column 1 from column 2, and subtract column 3 from column 4.

3. Remove the factor $(p_2-p_1)(\bar{p}_2-\bar{p}_1) = 1$ from the determinant as in equation (65).

4. Rearrange rows and columns so as to have the four vanishing elements occupy the upper right quadrant of the determinant.

Then the buckling condition becomes

$$\begin{vmatrix} \bar{Q} & Q & 0 & 0 \\ 1 & 1 & 0 & 0 \\ p_2\bar{Q} & \bar{p}_2 Q & -Q & -\bar{Q} \\ p_1^2 & \bar{p}_1^2 & (\bar{p}_1+\bar{p}_2) & (p_1+p_2) \end{vmatrix} = (\bar{Q}-Q)[\bar{Q}(\bar{p}_1+\bar{p}_2)-Q(p_1+p_2)] = 0 \tag{98}$$

The first factor in equation (98) vanishes for $\rho = 1$. From equation (79), the second factor vanishes if

$$\rho(r-4n^2+\rho)^{1/2}-(1-\rho^2)^{1/2}(r+4n^2-\rho)^{1/2} = 0 \tag{99}$$

If equation (99) is multiplied by $(r+4n^2-\rho)^{1/2}$ and the factor $(1-\rho^2)^{1/2}$ is disregarded as in equation (81), the following result is obtained:

$$r = 2\rho - 4n^2 \tag{100}$$

If both sides of this expression are squared and the definition of r is introduced, the quadratic equation

$$1 - 8n^2\rho + 16n^4 = (2\rho - 4n^2)^2 \tag{101}$$

is obtained; this result may also be written as

$$4\rho^2 - 8n^2\rho - 1 = 0 \tag{102}$$

The positive root of (102) is

$$\rho = \frac{1}{2}\sqrt{1+4n^4} + n^2 \tag{103}$$

which is very nearly one-half for thin shells.

FC2 CASE

Equations (56) and (61) will be satisfied at $\xi = 0$ if the first row of determinant in equation (97) is replaced by

$$\begin{bmatrix} p_2\bar{Q} & p_1\bar{Q} & \bar{p}_2 Q & \bar{p}_1 Q \end{bmatrix} \tag{104}$$

If a sequence of elementary operations is applied to the determinant, the buckling condition can be simplified to

$$r \begin{vmatrix} \bar{Q} & Q & 0 & 0 \\ 1 & 1 & 0 & 0 \\ p_2^2\bar{Q} & \bar{p}_2^2 Q & -Q & -\bar{Q} \\ p_1^2 & \bar{p}_1^2 & 1 & 1 \end{vmatrix} = r(\bar{Q}-Q)^2 = 0 \tag{105}$$

The factor r cannot vanish for values of ρ of interest, and $(\bar{Q}-Q)$ vanishes for $\rho = 1$; therefore, buckling at values of stress less than the classical value is not possible. A discussion of buckling at $\rho = 1$ is given in Appendix III.

FC3 CASE

Equations (56) and (62) will be satisfied at $\xi = 0$ for nontrivial values of A_1 through A_4 if

$$\begin{vmatrix} \bar{Q} & \bar{Q} & Q & Q \\ (1-\bar{Q}p_1^2) & (1-\bar{Q}p_2^2) & (1-Q\bar{p}_1^2) & (1-Q\bar{p}_2^2) \\ p_1 & p_2 & \bar{p}_1 & \bar{p}_2 \\ p_1^3 & p_2^3 & \bar{p}_1^3 & \bar{p}_2^3 \end{vmatrix} = 0 \tag{106}$$

The application of a sequence of elementary operations reduces the buckling condition to

$$-(\bar{Q}-Q)^2 + n^2[\bar{Q}(p_1+p_2)-Q(\bar{p}_1+\bar{p}_2)][Q(p_1+p_2)-\bar{Q}(\bar{p}_1+\bar{p}_2)] = 0 \tag{107}$$

If equations (78), (79), and (90) are utilized, the buckling condition simplifies to

$$2(1-\rho^2) - n^2[r(2\rho^2-1) - 4n^2 + \rho] = 0 \tag{108}$$

By neglecting terms involving n^4 in equation (108) and using equation (94), an approximate equation is obtained; this equation is

$$2(\rho+1)[1-\rho + n^2(\frac{1}{2} - \rho)] = 0 \tag{109}$$

The bracketed term in this expression vanishes (to the same approximation in terms up to n^4) when

$$\rho \approx 1 - \frac{n^2}{2} \tag{110}$$

which implies buckling at slightly less than the classical value of stress.

FC4 CASE

The FC4 case represents the most rigid condition of tangential restraint. Since buckling stress ratios of unity and slightly less than unity have been found for the FC2 and FC3 conditions of restraint, there is little doubt that buckling will occur at $\rho = 1$ in this case. It is indicated in Appendix III that $\rho = 1$ is always a possible buckling stress ratio for all cases of edge restraint; this fact, along with the intuitive physical consequences of edge fixity, leads to the conclusion that ρ should be unity in this instance.

SIMPLY SUPPORTED CASES

The SS1, SS2, and SS3 cases are treated in detail by Hoff and the author in Reference 4. The buckling stress ratios in the first two cases are the same as in the FC1 case, while the SS3-supported shell can buckle only at $\rho = 1$ (this is the classical, simply supported shell).

The same argument presented for the FC4 case applies equally well to the SS4 case. Since $\rho = 1$ is a possible buckling stress ratio for all cases and it is the only possible value for buckling in the SS3 case, where the tangential restraint is less severe, one must conclude that $\rho = 1$ is the buckling condition for SS4-supported shells.

COMPLETELY CLAMPED CASES

All the cases, CC1 through CC4, were solved numerically by Hoff and Soong [6]. An examination of their results indicates that if

$$\frac{L}{R} > 10 \frac{t}{R} \tag{111}$$

where L is the shell length, then ρ is essentially unity for all clamped edge conditions.

The results of Reference 6 can be verified readily, and the next section is devoted to a short analysis of the CC1 case.

CC1 CASE

The nontrivial satisfaction of equations (58) and (60) at $\xi = 0$ is assured if the following determinantal equation is satisfied:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \bar{Q} & \bar{Q} & Q & Q \\ p_1 & p_2 & \bar{p}_1 & \bar{p}_2 \\ p_1\bar{Q} & p_2\bar{Q} & \bar{p}_1 Q & \bar{p}_2 Q \end{vmatrix} = 0 \qquad (112)$$

The rows in the above determinant reflect the vanishing of w, $\sigma^0_{x_1}$, $w_{,\xi}$, and $\tau^0_{xy_1}$, respectively.

If column 1 is subtracted from column 2, column 3 is subtracted from column 4, the factor $(p_2 - p_1)(\bar{p}_2 - \bar{p}_1) = 1$ is noticed, and the columns are rearranged in the above determinant, equation (112) can be replaced by

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ \bar{Q} & Q & 0 & 0 \\ p_1 & \bar{p}_1 & 1 & 1 \\ p_1\bar{Q} & \bar{p}_1 Q & Q & \bar{Q} \end{vmatrix} = -(\bar{Q}-Q)^2 = 0 \qquad (113)$$

$(\bar{Q}-Q)$ vanishes only at $\rho = 1$, so buckling cannot occur at values of stress less than the classical value.

CC2, CC3, AND CC4 CASES

Each of these cases implies more rigid edge fixation than the CC1 case; it is realized, therefore, that $\rho = 1$ corresponds to buckling in these cases.

CONCLUDING REMARKS

A theoretical analysis, based upon linear theory, of the buckling behavior of axially compressed circular cylindrical shells has been undertaken to determine the effect of edge restraint upon the buckling stress; and overall evaluation of this effect has been made by studying the 16 limiting cases of restraint which correspond to the vanishing of generalized forces and generalized displacements at an edge. Of the 16 limiting cases, 8 permit buckling to occur at stresses less than the classical value of buckling stress.

The buckling stress ratios for all the cases are displayed in the table on page 28. While buckling in one case occurs at a value of stress only slightly less than the classical one, buckling stresses approaching one-half of this value are predicted in six instances which, interestingly enough, represent a rather wide variety of restraining conditions; a further 12 percent stress reduction has been found previously by Nachbar and Hoff for completely unrestrained cylinders of sufficient length. In addition to the seven cases of restraint which have been analyzed here for the first time, the work of previous investigators for the remaining cases has been reestablished and supplemented as well. Since the effect of shell length has been shown to be of minor importance for shells of practical interest, the analysis is simplified by treating semi-infinite shells, and this simplification permits solutions to be obtained in essentially closed form.

On the basis of the results obtained, both by previous investigators and by the present analysis, it is concluded that the mode of edge restraint can have a very significant effect upon the buckling stress of thin, axially compressed circular cylindrical shells. This effect has been widely overlooked in the past, a possible reason being the absence of a pronounced shell length effect upon the buckling stress in both theory and experiment except when extremely short shells are considered.

BUCKLING STRESS RATIOS FOR LIMITING CASES OF EDGE RESTRAINT

Normal Restraint / Tangential Restraint	FF: $R_x = M_x = 0$	FC: $R_x = w_{,\xi} = 0$	SS: $w = M_x = 0$ (Reference 4)	CC: $w = w_{,\xi} = 0$ (Reference 6)
1: $\sigma_{x_1} = 0$ $\tau_{xy_1} = 0$	$\rho = 0.380$ for $\nu = 0.3$, $n = 0.3$ (References 2 and 6)	$\rho = \frac{1}{2} + o(n^2)$	$\rho = \frac{1}{2} + o(n^2)$	$\rho = 1$
2: $u = 0$ $\tau_{xy_1} = 0$	$\rho = \frac{1}{2} + o(n^2)$	$\rho = 1$	$\rho = \frac{1}{2} + o(n^2)$	$\rho = 1$
3: $\sigma_{x_1} = 0$ $v = 0$	$\rho = \frac{1}{2} + o(n^2)$	$\rho = 1 - o(n^2)$	$\rho = 1$	$\rho = 1$
4: $u = 0$ $v = 0$	$\rho = \frac{1}{2} + o(n^2)$	$\rho = 1$	$\rho = 1$	$\rho = 1$

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APPENDIX I

PROOF THAT THE ROOTS p_1 THROUGH p_4 HAVE NEGATIVE REAL PARTS WHEN $\rho < 1$

The roots p_1 through p_4 will have negative real parts if the parameters α_1 and α_2 defined in equation (44) are negative; these parameters are

$$\alpha_1 = -(1-\rho)^{1/2} - (r+4n^2-\rho)^{1/2} \tag{114}$$

$$\alpha_2 = (1-\rho)^{1/2} - (r+4n^2-\rho)^{1/2} \tag{115}$$

If $\rho < 1$, α_1 will be negative if

$$r + 4n^2 - \rho > 0 \tag{116}$$

By the definition in equation (41) for r, equation (116) can be written

$$[(4n^2-\rho)^2 + 1-\rho^2]^{1/2} + (4n^2-\rho) > 0, \tag{117}$$

which proves that α_1 is obviously negative for $\rho < 1$.

Equation (115) implies that α_2 will be negative if

$$(r+4n^2-\rho)^{1/2} > (1-\rho)^{1/2} \tag{118}$$

This is equivalent to the requirement that

$$r > 1 - n^2 \tag{119}$$

Now r can be conveniently written as

$$r = [(4n^2-\rho)^2 + 1-\rho^2]^{1/2} = [(1-4n^2)^2 + 8n^2(1-\rho)]^{1/2} \tag{120}$$

Thus, if equation (120) is substituted into equation (119), it is clear that α_2 is also negative for $\rho < 1$.

These conclusions are stated in Reference 14, but a proof is not included. A less direct proof is also given in Reference 4.

APPENDIX II

IDENTITIES INVOLVING THE ROOTS p_1, p_2, p_3, AND p_4

Equations (28) through (41) define and describe the roots of equation (24). The four roots of interest, which possess negative real parts for $\rho < 1$, can be written as

$$p_1 = \frac{1}{2}[-Q^{1/2} - (Q+4n^2)^{1/2}] \tag{121}$$

$$p_2 = \frac{1}{2}[Q^{1/2} - (Q+4n^2)^{1/2}] \tag{122}$$

$$\bar{p}_1 = \frac{1}{2}[-\bar{Q}^{1/2} - (\bar{Q}+4n^2)^{1/2}] = p_3 \tag{123}$$

$$\bar{p}_2 = \frac{1}{2}[\bar{Q}^{1/2} - (\bar{Q}+4n^2)^{1/2}] = p_4 \tag{124}$$

These roots satisfy the following equations:

$$\left.\begin{aligned} \frac{p_1^2-n^2}{p_1} &= -Q^{1/2}, & \frac{p_2^2-n^2}{p_2} &= Q^{1/2} \\ \frac{\bar{p}_1^2-n^2}{\bar{p}_1} &= -\bar{Q}^{1/2}, & \frac{\bar{p}_2^2-n^2}{\bar{p}_2} &= \bar{Q}^{1/2} \end{aligned}\right\} \tag{125-128}$$

On the basis of equations (121) through (124), it is clear that

$$\left.\begin{aligned} p_2 - p_1 &= Q^{1/2} & p_1 + p_2 &= -(Q+4n^2)^{1/2} \\ \bar{p}_2 - \bar{p}_1 &= \bar{Q}^{1/2} & \bar{p}_1 + \bar{p}_2 &= -(\bar{Q}+4n^2)^{1/2} \\ & (p_2-p_1)(\bar{p}_2-\bar{p}_1) = Q^{1/2}\bar{Q}^{1/2} = 1 \end{aligned}\right\} \tag{129-133}$$

Equations (125) through (128) provide the identities

$$p_1^2 + p_2^2 = 2n^2 + (p_2-p_1)Q^{1/2} = 2n^2 + Q \tag{134}$$

$$\bar{p}_1^2 + \bar{p}_2^2 = 2n^2 + (\bar{p}_2-\bar{p}_1)\bar{Q}^{1/2} = 2n^2 + \bar{Q} \tag{135}$$

With the aid of equations (40) and (41), it is clear that

$$(p_1 + p_2)(\overline{p}_1 + \overline{p}_2) = \left|(Q+4n^2)^{1/2}\right|^2 = r \tag{136}$$

The identities given here, in addition to those derived in the development of the solutions, are sufficient for the derivation of the buckling solutions. Some of the above identities were originally given by Nachbar in Reference 14, and others can be found in References 4 and 6.

APPENDIX III

PROOF THAT $\rho = 1$ IS ALWAYS A BUCKLING STRESS RATIO FOR LONG SHELLS

By assigning the value unity to the stress ratio ρ, equations (28), (29), (31), and (32) yield the following simple expressions:

$$Q = \bar{Q} = -1 \tag{137}$$

$$Q^{1/2} = i \tag{138}$$

$$\bar{Q}^{1/2} = -i \tag{139}$$

If these results are substituted into equations (121) through (124), it is found that

$$p_1 = -\frac{i}{2}[1 + (1-4n^2)^{1/2}] \tag{140}$$

$$p_2 = \frac{i}{2}[1 - (1-4n^2)^{1/2}] \tag{141}$$

$$p_3 = p_2 \tag{142}$$

$$p_4 = p_1 \tag{143}$$

Equations (137), (142), and (143) imply that the first and fourth columns, as well as the second and third columns, of all the buckling determinants written for semi-infinite shells become identical when $\rho = 1$. Therefore, the vanishing of all these determinants at $\rho = 1$ is assured.

The question of whether buckling modes for $\rho = 1$ can be found to satisfy a given set of edge restraint conditions is rather academic. It is sufficient merely to refer to the numerical results obtained by Hoff and Soong in Reference 6 for shells of finite length. These results indicate that even for a CC4-supported shell of finite length, ρ is essentially unity whenever

$$\frac{L}{R} > 10\,\frac{t}{R}$$

where L is the length of the shell. If $t/R = 1/100$, then a shell must be shorter than its radius before the buckling stress is noticeably increased above the classical value. Since the CC4 case represents the most severe condition of edge restraint, unity is an upper bound for the buckling stress ratio which can be applied to all shells whose lengths exceed their respective radii, regardless of the condition of edge restraint.

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13. ABSTRACT

A broad analytical study of the effect of edge restraint on the buckling stress of geometrically perfect, axially compressed circular cylindrical shells, based on linear elastic theory, is presented. An overall evaluation of this effect is made by studying the 16 limiting cases of restraint which correspond to the vanishing of generalized forces and generalized displacements at an edge. Of the 16 limiting cases, 7 of which are studied for the first time, it is found that 8 permit buckling to occur at stresses less than the classical theoretical value of buckling stress. The fact that both theory and experiments indicate that the effect of shell length on the buckling stress can usually be disregarded is used in the analysis. This fact suggests that shells may be assumed to be semi-infinite; as a consequence, formulas for buckling stresses are determined in essentially closed form.

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